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SUBELLIPTIC ESTIMATES FOR THE $\bar{\partial}$-NEUMANN PROBLEM IN C^2

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1. Introduction

In this paper we prove a conjecture of J. J. Kohn concerning precise subelliptic estimates for the local $\bar{\partial}$-Neumann problem in C^2. Let Ω be a bounded open set in C^2 with C^∞ boundary ω. If ω is pseudoconvex near a point $P \in \omega$, and P is of type m (the precise definitions are given in § 2), then Kohn proved that the subelliptic esitimate

$$\|\phi\|_{(s)}^{(\Omega)} \leq C_s(\|\bar{\partial}\phi\|_{(0)}^{(\Omega)} + \|\theta\phi\|_{(0)}^{(\Omega)} + \|\phi\|_{(0)}^{(\Omega)})$$

holds for all $s < 1/(m+1)$ (see [8, (7.4)]). Here ϕ is a C^∞ one-form with compact support in $\Omega \cap U$ where U is some sufficiently small neighborhood of P, θ is the adjoint of $\bar{\partial}$, and ϕ is in the domain of θ.

In [7] and [8] Kohn suggested that (i) the subelliptic estimate in question holds with $s = 1/(m+1)$, and (ii) it cannot hold with $s > 1/(m+1)$. In Theorem 3.7 of this paper we shall prove the second conjecture. We do not know whether $s = 1/(m+1)$ is achieved. In proving Theorem 3.7 we make use of results obtained by Yu. V. Egorov [1], L. Hörmander [4], [5] and W. J. Sweeney [9], which enable us to reduce the problem to a similar question concerning a system of pseudo-differential operators on ω. We shall compute these pseudo-differential operators with great precision by utilizing some results of Kohn (see [7] and [8]) concerning the behavior of ω near a point of type m. Our notation and terminology are standard (see e.g. [3] and [4]).

2. The Levi invariants

We recall the basic definitions of [8]. Let Ω be a bounded open subset of C^2 with C^∞ boundary ω, and let $r(P)$ denote the distance of the point P from ω, and assume that $r < 0$ in Ω and $r > 0$ outside of Ω. A vector field L is said to be *holomorphic* in some open set $U \subset C^2$ if it can be written in the form

$$L = a^1 \frac{\partial}{\partial z_1} + a^2 \frac{\partial}{\partial z^2}\,, \qquad a^i \in C^\infty(U)\,, \tag{2.1}$$

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where $\partial/\partial z_j = \frac{1}{2}(\partial/\partial x_j - i(\partial/\partial y_j))$, $j = 1, 2$. A vector field L is said to be *tangential* if at each point of ω it is tangent to ω, that is, if $L(r) = 0$ at $r = 0$. As usual we define $\bar{L}$ by

$$\bar{L} = \bar{a}^1 \frac{\partial}{\partial \bar{z}_1} + \bar{a}^2 \frac{\partial}{\partial \bar{z}_2} . \tag{2.2}$$

If T_1 and T_2 are two vector fields, we define the Lie bracket by $[T_1, T_2] = T_1T_2 - T_2T_1$. The Lie algebra generated by T_1 and T_2 over the C^∞ functions is the smallest module over the C^∞ functions closed under $[\ ,\ ]$, and is denoted by $\mathscr{L}\{T_1, T_2\}$. $\mathscr{L}\{T_1, T_2\}$ is filtered, that is,

$$\mathscr{L}\{T_1, T_2\} = \bigcup_{k=0}^{\infty} \mathscr{L}_k\{T_1, T_2\} ,$$

where $\mathscr{L}_0\{T_1, T_2\}$ is the module spanned by T_1 and T_2, and $\mathscr{L}_{k+1}\{T_1, T_2\}$ is the module spanned by the elements of $\mathscr{L}_k\{T_1, T_2\}$ and the elements of the form $[A, T_i]$ with $A \in \mathscr{L}_k\{T_1, T_2\}$. Set

$$\mathscr{L} = \mathscr{L}\{L, \bar{L}\} , \qquad \mathscr{L}_k = \mathscr{L}_k\{L, \bar{L}\} ,$$

where L is a holomorphic tangent vector in some neighborhood of a point $P \in \omega$, which is different from zero at P. We note that the $\mathscr{L}$ and $\mathscr{L}_k$ evaluated at P do not depend on the choice of L.

2.3. Definition. $P \in \omega$ is said to be of finite type if there exists $F \in \mathscr{L}$ such that $\langle (\partial r)_P, F_P \rangle \neq 0$. Here $\langle\ ,\ \rangle$ denotes contraction between cotangent vectors and tangent vectors, and the subscript P denotes evaluation at P. P of finite type is said to be of *type m* if m is the least integer such that there is an element in $\mathscr{L}_m$ satisfying the above property.

2.4. Definition. Ω is said to be pseudo-convex near a point $P \in \omega$ if there is a neighborhood U of P such that

$$\langle \partial r, [\bar{L}, L] \rangle_{\omega \cap U} \geq 0 , \tag{2.5}$$

where L is a nonzero tangential holomorphic vector field.

2.6. Definition. If Ω is pseudo-convex near a point $P \in \omega$, and P is of type m, we say that ω is pseudo-convex of order m at P.

3. The local $\bar{\partial}$-Neumann problem in C^2

Let $H_{(s)}^{(\Omega)}$ and $H_{(s)}^{(\omega)}$ denote the Sobolev spaces on Ω and ω respectively (see e.g. [3]) with norms denoted by $\|\ \|_{(s)}^{(\Omega)}$ and $\|\ \|_{(s)}^{(\omega)}$ as usual. These spaces and norms are well defined for vector functions, in particular, for (0, 1)-forms $\phi = \phi_1 d\bar{z}_1 + \phi_2 d\bar{z}_2, \phi_1, \phi_2 \in C^\infty(\Omega)$. On (0, 1)-forms we have

$$\bar{\partial}\phi = (\partial\phi_2/\partial\bar{z}_1 - \partial\phi_1/\partial\bar{z}_2)d\bar{z}_1 \wedge d\bar{z}_2 . \tag{3.1}$$

Let θ denote the formal adjoint of $\bar{\partial}$ operating on (0, 1)-forms, that is,

$$(\bar{\partial}\phi, \psi)_{L^2(\Omega)} = (\phi, \theta\psi)_{L^2(\Omega)} , \tag{3.2}$$

$\phi \in C_0^\infty(\Omega)$ and $\psi \in D_{(0,1)}(\Omega)$, where $D_{(0,1)}(\Omega)$ stands for C^∞ (0, 1)-forms with compact support in Ω. More precisely we have

$$\theta(\phi_1 d\bar{z}_1 + \phi_2 d\bar{z}_2) = - \partial\phi_1/\partial z_1 - \partial\phi_2/\partial z_2 . \tag{3.3}$$

Now we can state the main result of [8].

3.4. Theorem. *Let $P \in \omega$ be a point of type m, and U be an open neighborhood of P such that $U \cap \omega$ is pseudoconvex. Then there exists a constant C_s for all $s, 0 < s < 1/(m + 1)$, such that*

$$\|\phi\|_{(s)}^{(\Omega)} \leq C_s(\|\bar{\partial}\phi\|_{(0)}^{(\Omega)} + \|\theta\phi\|_{(0)}^{(\Omega)} + \|\phi\|_{(0)}^{(\Omega)}) \tag{3.5}$$

for all $\phi \in D_{(0,1)}(U \cap \bar{\Omega})$ satisfying $\langle\phi, \bar{\partial}r\rangle = 0$ on $\omega \cap U$.

We note that $\langle\psi, \bar{\partial}r\rangle = 0$ on $\omega \cap U$ is equivalent to

$$\langle\bar{\partial}\phi, \psi\rangle = \langle\phi, \theta\psi\rangle , \qquad \phi \in D_{(0,1)}(U \cap \bar{\Omega}) .$$

When $m = 1$, (3.4) holds with $s = \frac{1}{2}$, and this is the best possible estimate (see [4], [6] and [10]). When $m > 1$, we do not have such a precise result. On the other hand, we have the following result.

3.7. Theorem. *Let $P \in \omega$ be a point of type m, and U a neighborhood of P. Then the estimate* (3.5) *does not hold with any $s > 1/(m + 1)$.*

The proof of Theorem 3.7 will be given in §§ 4, 5 and 6.

4. The $\bar{\partial}$ operator near a point of type m

Let $P \in \omega$ be a point of type m, and U a sufficiently small neighborhood of P. By an affine change of coordinates we construct coordinates z_1', z_2' in U such that

$$z_1'(P) = z_2'(P) = (\partial r/\partial z_1')_P = (\partial r/\partial \bar{z}_1')_P = (\partial r/\partial y_2')_P = 0 , \tag{4.1}$$

$$(\partial r/\partial x_2')_P = 1 ,$$

where $z_1' = x_1' + iy_1'$ and $z_2' = x_2' + iy_2'$. Now r has the following Taylor series expansion

$$r(z') = \operatorname{Re} h(z') + \psi(z') + O(|z'|^{m+2}) , \tag{4.2}$$

where $\psi(z')$ is a polynomial of degree $m + 1$ such that each term contains $z_i'\bar{z}_j'$ as a factor and

$$h(z_1', z_2') = \sum_{s+t \le m+1} \frac{1}{s!\,t!} \{(\partial/\partial z_1')^s (\partial/\partial z_2')^t r\} z_1'^s z_2'^t \,. \tag{4.3}$$

According to (4.1) and (4.2)

$$(\partial h/\partial z_1')_0 = 0 \,, \qquad (\partial h/\partial z_2')_0 = (\partial r/\partial x_2')_0 = 1 \,. \tag{4.4}$$

Thus z_1' and h are linearly independent in U (here we need U to be sufficiently small), and therefore we can introduce holomorphic coordinates $w_1 = u_1 + iv_1$, $w_2 = u_2 + iv_2$ defined by $w_1 = z_1'$ and $w_2 = h$. Then (4.2) becomes

$$r(w_1, w_2) = u_2 + \gamma(w_1, w_2) + O(|w|^{m+2}) \,, \tag{4.5}$$

where

$$\gamma(w_1, w_2) = O(|w|^2) \tag{4.6}$$

is a polynomial of degree $m + 1$ which contains no pure terms, that is, holomorphic or antiholomorphic terms.

To derive a precise expression for the $\bar\partial$ operator we set

$$|\nabla r|\omega^1 = r_{\bar w_2}\, dw_1 - r_{\bar w_1}\, dw_2 \,, \qquad |\nabla r|\omega^2 = r_{w_1}\, dw_1 + r_{w_2}\, dw_2 = \partial r \,,$$

where $r_{w_1} = \partial r/\partial w_1$, etc., so that ω_1 and ω_2 yield a basis of the (1,0)-forms in U. Let $\phi = \phi_1\bar\omega^1 + \phi_2\bar\omega^2$. From (3.1) it is easy to see that the $\bar\partial$ operator on (0,1)-forms ϕ has the following expression in terms of the basis $\bar\omega^1$ and $\bar\omega^2$:

$$\begin{aligned}\bar\partial\phi = {} & (-\bar M\phi_1 + \bar L\phi_2)\bar\omega^1 \wedge \bar\omega^2 \\ & + (\text{terms in which } \phi_1 \text{ and } \phi_2 \text{ remain undifferentiated}) \,,\end{aligned} \tag{4.7}$$

where

$$|\nabla r| L = r_{w_2} \frac{\partial}{\partial w_1} - r_{w_1} \frac{\partial}{\partial w_2} \,, \tag{4.8}$$

$$|\nabla r| M = r_{\bar w_1} \frac{\partial}{\partial w_1} + r_{\bar w_2} \frac{\partial}{\partial w_2} \,. \tag{4.9}$$

Given $\phi = \phi_1\bar\omega^1 + \phi_2\bar\omega^2$ the $\bar\partial$-Neumann boundary condition $\langle \phi, \bar\partial r\rangle = 0$ on ω is equivalent to the vanishing of ϕ_2 on ω. If $\phi = \phi_1\bar\omega^1 + \phi_2\bar\omega^2 \in C_0^\infty(U \cap \bar\Omega)$ and $\phi_2 = 0$ on ω, then $\theta\phi$ is well defined and is given by the expression

$$\begin{aligned}\theta\phi = {} & -(L\phi_1 + M\phi_2) \\ & + (\text{terms in which } \phi_1 \text{ and } \phi_2 \text{ remain undifferentiated}).\end{aligned} \tag{4.10}$$

Thus in terms of the basis $\bar{\omega}^1, \bar{\omega}^2$ the principal part of the $\bar{\partial}$-Neumann operator on (0,1)-forms is given by

$$D_0 = \begin{pmatrix} -\bar{M} & \bar{L} \\ -L & -M \end{pmatrix} . \tag{4.11}$$

4.12. Lemma. *Let $P \in \omega$ be of type m. Then $\gamma(w_1, 0)$ is a homogeneous polynomial in w_1 of degree $m+1$. More precisely*

$$\gamma(w_1, 0) = \sum_{s+t=m-1} \frac{1}{(s+1)!\,(t+1)!} L^s \bar{L}^t \langle \partial r, [L, \bar{L}] \rangle w_1^{s+1} \bar{w}_1^{t+1} . \tag{4.13}$$

Proof. See Kohn [8, Lemma 3.16].

Consider

$$r = u_2 + \gamma(u_1, v_1, u_2, v_2) + O(|u|^{m+2} + |v|^{m+2}) = 0 , \tag{4.14}$$

where we set $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Since $(\partial r/\partial u_2)_0 = 1$, we can solve (4.14) for $u_2 = u_2(u_1, v_1, v_2)$ in a neighborhood of 0.

4.15. Lemma. *Let $u_2(u_1, v_1, v_2)$ be a solution of* (4.14) *in some neighborhood of* 0. *Then*

$$\frac{\partial^{l+k} u_2(0)}{\partial u_1^l \partial v_1^k} = 0 \quad \text{if} \quad l + k \le m . \tag{4.16}$$

Proof. According to Lemma 4.12

$$\frac{\partial^{l+k} \gamma(0)}{\partial u_1^l \partial v_1^k} = 0 \quad \text{if} \quad l + k \le m . \tag{4.17}$$

By the definition of r, $u_2(0) = 0$. Next, replacing u_2 by $u_2(u_1, v_1, v_2)$ in (4.17) we obtain

$$\frac{\partial u_2}{\partial u_1} + \frac{\partial \gamma}{\partial u_1} + \frac{\partial \gamma}{\partial u_2}\frac{\partial u_2}{\partial u_1} + O(|u_1|^{m+1} + |v|^{m+1}) = 0 .$$

Since $\gamma = O(|u|^2 + |v|^2)$, this implies that

$$\frac{\partial u_2(0)}{\partial u_1} = 0, \text{ and similarly } \frac{\partial u_2(0)}{\partial v_1} = 0 . \tag{4.18}$$

Now suppose that

$$\frac{\partial^{l+k} u_2(0)}{\partial u_1^l \partial v_1^k} = 0 \quad \text{if} \quad l + k \le p \tag{4.19}$$

for some $p < m$. Then for a fixed l and k satisfying $l + k = p + 1$ we have

$$\frac{\partial^{p+1}u_2}{\partial u_1^l \partial v_1^k} + \frac{\partial^{p+1}\gamma}{\partial u_1^l \partial v_1^k} + \sum_{\sum_j (s_j+t_j+(q_j-1)) \leq p+1} C_{\{s_j,t_j,q_j\}_j} \prod_j \left(\frac{\partial^{s_j+t_j}u_2(0)}{\partial u_1^{s_j}\partial v_1^{t_j}}\right)^{q_j} + O((|u_1| + |v|)^{m+1-p}) = 0 .$$

In particular if we set $u_1 = v_1 = v_2 = 0$, the induction hypothesis (4.19) implies that

$$\frac{\partial^{p+1}u_2(0)}{\partial u_1^l \partial v_1^k} + \frac{\partial\gamma(0)}{\partial u_2}\frac{\partial^{p+1}u_2(0)}{\partial u_1^l \partial v_1^k} = \frac{\partial^{p+1}u_2(0)}{\partial u_1^l \partial v_1^k} = 0 .$$

This proves Lemma 4.15.

To utilize Lemma 4.15 we set $x_1 = u_1$, $x_2 = v_1$, $x_3 = v_2$ and $\rho = -r$. Then a simple computation yields

$$(4.20) \qquad |\nabla\rho| L = -\frac{1}{2}\rho_{w_2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) - \frac{1}{2}i\rho_{w_1}\frac{\partial}{\partial x_3} ,$$

$$(4.21) \qquad |\nabla\rho| M = -|\nabla_w\rho|^2\frac{\partial}{\partial\rho} - \frac{1}{2}\rho_{\bar{w}_1}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right) + \frac{1}{2}i\rho_{w_2}\frac{\partial}{\partial x_3} ,$$

where

$$(4.22) \qquad |\nabla_w\rho|^2 = |\rho_{w_1}|^2 + |\rho_{w_2}|^2 .$$

5. Reduction to the boundary

In [4] L. Hörmander reduced the study of the estimate (3.5) from $U \cap \Omega$ to the study of similar estimates involving pseudo-differential operators on $U \cap \omega$, at least in the case $s = \frac{1}{2}$. This result was extended by W. J. Sweeney [10] to arbitrary s, $0 < s \leq 1$. To be able to state the result in our particular case we shall first compute the boundary system of pseudo-differential operators in question. From (4.11) we have

$$(5.1) \qquad D_0^* D_0 = (L(-\bar{L}) + M(-\bar{M}))I_2 + \text{first order terms} ,$$

where I_2 stands for the two-by-two identity matrix. Let r^0 denote $d^{0*}d^0$, the principal symbol of $D_0^* D_0$. A somewhat messy calculation yields

$$(5.2) \qquad \begin{aligned} r^0(x,\xi,\tau) &= (|L(x,\xi)|^2 + |M(x,\xi,\tau)|^2)I_2 \\ &= \tfrac{1}{4}\{|\nabla_w\rho|^2\tau^2 + [\mathrm{Re}(\rho_{w_1}(\xi_1 - i\xi_2)) \\ &\quad - (\mathrm{Im}\,\rho_{w_2})\xi_3]\tau + \tfrac{1}{4}|\xi|^2\}I_2|\nabla\rho|^{-2} , \end{aligned}$$

where τ stands for the symbol of $\partial/i\partial\rho$, and ρ is assumed to be zero. The equation $r^0(x, \xi, D_\rho)U(\rho) = 0$ has a unique exponentially decreasing solution on R_+ such that $U(0) = u$, which is given by

$$\begin{pmatrix} u_1 & e^{m\rho} \\ u_2 & e^{m\rho} \end{pmatrix} \tag{5.3}$$

where

$$\begin{aligned} m = \tfrac{1}{2}|\nabla_m\rho|^{-2}\{&- i[\mathrm{Re}(\rho_{\bar{w}_1}(\xi_1 - i\xi_2)) - (\mathrm{Im}\,\rho_{w_2})\xi_3] \\ &- (|\nabla_w\rho|^2|\xi|^2 - [\mathrm{Re}(\rho_{\bar{w}_1}(\xi_1 - i\xi_2)) - (\mathrm{Im}\,\rho_{w_2})\xi_3]^2)^{\frac{1}{2}}\}\ . \end{aligned} \tag{5.4}$$

Following Hörmander (see [4, Theorem 2.3.1]) we define pseudo-differential operators P_1 and P_2 on $U \cap \omega$ with principal symbols $p_1^0(x, \xi)$ and $p_2^0(x, \xi)$, respectively, given by the first column of

$$d^0(x, \rho = 0, \xi, D_\rho)\begin{pmatrix} e^{m\rho} \\ 0 \end{pmatrix} \tag{5.5}$$

evaluated at $\rho = 0$. More explicitly we have

$$\begin{aligned} p_1^0(x, \xi) = \tfrac{1}{2}\,&\mathrm{Im}(\rho_{\bar{w}_1}(\xi_1 - i\xi_2)) - \tfrac{1}{2}(\mathrm{Re}\,\rho_{w_2})\xi_3 \\ &- \tfrac{1}{2}\{|\nabla_w\rho|^2|\xi|^2 - [\mathrm{Re}(\rho_{\bar{w}_1}(\xi_1 - i\xi_2)) - \mathrm{Im}(\rho_{w_2})\xi_3]^2\}^{\frac{1}{2}}\ , \end{aligned} \tag{5.6}$$

$$p_2^0(x, \xi) = -\tfrac{1}{2}\, i\rho_{w_2}(\xi_1 - i\xi_2) + \tfrac{1}{2}\,\rho_{w_1}\xi_3\ . \tag{5.7}$$

5.8. Proposition. *Let* $0 < s \le 1$. *Then* (3.5) *implies the following estimate*

$$\|\phi\|_{(s)}^{(\omega)} \le C_s(\|P_1\phi\|_{(0)}^{(\omega)} + \|P_2\phi\|_{(0)}^{(\omega)} + \|\phi\|_{(0)}^{(\omega)}) \tag{5.9}$$

for all $\phi \in C_0^\infty(U \cap \omega)$.

Proof. Recall that the $\bar{\partial}$-Neumann boundary condition is equivalent to $\phi_2 = 0$ on ω. Then Proposition 5.8 is a special case of the results of Hörmander (see [4, Theorems 2.3.1 and 2.3.2]) and of Sweeney (see [10, Propositions 5.7 and 5.8]).

6. Proof of Theorem 3.7

First we localize the estimate (5.9).

6.1. Proposition. *Let* $0 < s \le 1$, *and set* $\delta = 1 - s$. *Suppose that the estimate* (5.9) *holds with*

(6.2)
$$\frac{k}{k+1} \leq \delta < \frac{k+1}{k+2},$$

where k is a positive integer. Then for every $(x, \xi) \in T^*(\omega)$, $|\xi| = 1$, there exists a constant C such that

(6.3)
$$\begin{aligned}\int_{R_3} |\phi(y)|^2 dy \\ \leq C\Big\{ \sum_{j=1,2} \int_{R_3} \Big| \sum_{|\alpha+\beta| \leq k} \frac{1}{\alpha!\,\beta!} \frac{\partial^{\alpha+\beta} p_j^0(x, \xi)}{\partial\xi^\alpha \partial x^\beta} y^\beta (D^\alpha \phi)(y) \lambda^{\delta - |\alpha|\delta - (1-\delta)|\beta|} \Big|^2 dy \\ + \lambda^{2\delta - 2(k+1)(1-\delta)} \sum_{|\alpha+\beta| \leq k+1} \int_{R_3} |y^\beta (D^\alpha \phi)(y)|^2 dy \lambda^{-2|\alpha|(2\delta-1)} \Big\},\end{aligned}$$

for all $\lambda \geq 1$ and $\phi \in C_0^\infty(R_3)$.

Proof. See Egorov [1, Theorem 1] and Hörmander [5, Theorems 6.1 and 6.3].

Let $x = 0$ be a point of type m. We shall show that the estimate (6.3) cannot hold at the point $(x_0, \xi^0) = (0, 0, 0, 0, 0, 1)$ when $k < m$. According to Proposition 6.1 this proves that the estimate (5.9) does not hold with $s > 1/(m+1)$, which proves Theorem 3.7. Since $\rho_{w_2}(0) = -\frac{1}{2}$, according to (5.6) and (5.7) we have

(6.4)
$$p_1^0(x_0, \xi^0) = p_2^0(x_0, \xi^0) = 0,$$

and therefore we can assume that $k \geq 1$. Furthermore Lemmas 4.12 and 4.15 imply that

(6.5)
$$\rho_{w_1}(x) = x_3 h(x) + O(|x|^m).$$

Assume that the estimate (6.3) holds for some δ such that $k/(k+1) \leq \delta < (k+1)/(k+2)$ with $k < m$. We substitute

(6.6)
$$\phi(y) = \psi(y_1, y_2, y_3 \lambda^{2\delta-1+\varepsilon}), \qquad \psi \in C_0^\infty(R_3)$$

into (6.3) with some $\varepsilon > 0$ such that

(6.7)
$$(k+1)\varepsilon + \delta < (k+1)(1-\delta) \leq 1.$$

According to the right hand side of (6.2) we have $\delta < (k+1)(1-\delta)$ so that such an ε can always be found. We change coordinates $y_1 = y_1'$, $y_2 = y_2'$, $y_3 \lambda^{2\delta-1+\varepsilon} = y_3'$, divide both sides of (6.3) by $\lambda^{-2\delta+1-\varepsilon}$, and let $\lambda \to \infty$. Then the left hand side of (6.3) becomes

(6.8)
$$\int_{R_3} |\psi(y')|^2 dy'.$$

Next we compute the right hand side of (6.3).

1) Terms involving $p_1^0(x, \xi)$ and its derivatives at (x_0, ξ^0).
(i) Set $q_1(x, \xi) = \mathrm{Im}(\rho_{w_1}(\xi_1 - i\xi_2))$. Then

$$\begin{aligned}
&\sum_{\substack{|\beta|\le k-1\\ j=1,2}} \frac{1}{\beta!} \frac{\partial^{\beta+1} q_1(x_0, \xi^0)}{\partial x^\beta \partial \xi_j} y^\beta (D_j\phi)(y) \lambda^{-(1-\delta)|\beta|} \\
&= \sum_{\substack{|\beta|\le k-1\\ j=1,2}} \frac{1}{\beta!} \frac{\partial^{\beta+1} q_1(x_0, \xi^0)}{\partial x^\beta \partial \xi_j} y'^\beta (D_j\psi)(y') \lambda^{-(1-\delta)|\beta| + \beta_3(-2\delta+1-\varepsilon)} \qquad (6.9)\\
&= O(\lambda^{-(1-\delta)}) ,
\end{aligned}$$

since $\beta_3 > 0$ by (6.5).
(ii) Set $p_1 = \frac{1}{2}q_1 - \frac{1}{2}q_2$. Then

$$\begin{aligned}
&\sum_{\substack{|\alpha+\beta|\le k\\ \alpha_1+\alpha_2\ne 0}} \frac{1}{\alpha!\beta!} \frac{\partial^{\alpha+\beta} q_2(x_0, \xi^0)}{\partial \xi^\alpha \partial x^\beta} y^\beta (D^\alpha\phi)(y) \lambda^{\delta-|\alpha|\delta-(1-\delta)|\beta|} \\
&= \sum_{\substack{|\alpha+\beta|\le k\\ \alpha_1+\alpha_2\ne 0}} \frac{1}{\alpha!\beta!} \frac{\partial^{\alpha+\beta} q_2(x_0, \xi^0)}{\partial \xi^\alpha \partial x^\beta} y'^\beta (D^\alpha\psi)(y') \qquad (6.10)\\
&\qquad \cdot \lambda^{-(\alpha_1+\alpha_2-1)\delta-\alpha_3(1-\delta-\varepsilon)-(1-\delta)(\beta_1+\beta_2)-\beta_3(\delta+\varepsilon)} \\
&= \sum_{j=1}^{2} \frac{\partial q_2(x_0, \xi^0)}{\partial \xi_j} (D_j\psi)(y') + o(1) = o(1) ,
\end{aligned}$$

because

$$\frac{\partial q_2(x_0, \xi^0)}{\partial \xi_j} = \left(\frac{\partial}{\partial \xi_j}\left(-\frac{1}{2} + \frac{1}{2}\sqrt{\xi_j^2+1}\right)\right)_{\xi_j=0} = 0 , \qquad j = 1, 2 .$$

(iii) Finally set $\xi_1 = \xi_2 = 0$ in q_2. Then

$$\begin{aligned}
q_2(x, 0, 0, \xi_3) &= ((|\rho_{w_1}|^2 + (\mathrm{Re}\,\rho_{w_2})^2)^{\frac{1}{2}} + \mathrm{Re}\,\rho_{w_2})\xi_3 \\
&= -\mathrm{Re}\,\rho_{w_2}\{1 - (1 + |\rho_{w_1}|^2/(\mathrm{Re}\,\rho_{w_2})^2)^{\frac{1}{2}}\}\xi_3 \\
&= \left(-(\mathrm{Re}\,\rho_{w_2}) \sum_{j=1}^{m} a_j \frac{|\rho_{w_1}|^{2j}}{(\mathrm{Re}\,\rho_{w_2})^{2j}} + O(|x|^{2m+2})\right)\xi_3 \qquad (6.11)\\
&= (x_3^2 H(x) + O(|x|^{2m+2}))\xi_3 ,
\end{aligned}$$

where $a_j, j = 1, \cdots, m$, are the coefficients in the Taylor series expansion $1 + \sum_{j=1}^m a_j x^j + O(|x|^{m+1})$ of $\sqrt{1+x}$ about $x = 0$, and $H(x)$ is a C^∞ function near $x = 0$. Thus

$$\begin{aligned}
&\sum_{|\beta+\alpha_3|\le k} \frac{1}{\beta!} \frac{\partial^{\beta+\alpha_3} q_2(x_0, \xi^0)}{\partial \xi_3^{\alpha_3} \partial x^\beta} y^\beta (D^{\alpha_3}\phi)(y) \lambda^{\delta-\alpha_3\delta-(1-\delta)|\beta|} \\
&= \sum_{|\beta|\le k} \frac{1}{\beta!} \frac{\partial^\beta q_2(x_0, \xi^0)}{\partial x^\beta} y'^\beta \psi(y') \lambda^{\delta+2(1-2\delta-\varepsilon)-(1-\delta)|\beta|+(\beta_3-2)(1-2\delta-\varepsilon)} \qquad (6.12)
\end{aligned}$$

$$+\sum_{|\beta|\le k-1}\frac{1}{\beta!}\frac{\partial^{\beta+1}q_2(x_0,\xi^0)}{\partial\xi_3\partial x^\beta}y'^\beta(D_3\psi)(y')\lambda^{-(1-\delta)|\beta|+(\beta_3-1)(1-2\delta-\varepsilon)}$$
$$=O(\lambda^{-(\delta+2\varepsilon)}),$$

since $\beta_3\ge 2$ according to (6.11).

2) As for $p_2^0(x,\xi)$ we have

$$-\tfrac{1}{4}(\partial\psi/\partial y_1-i\partial\psi/\partial y_2)$$
$$-\frac{1}{2}i\sum_{0<|\beta|\le k-1}\frac{1}{\beta!}\frac{\partial^\beta\rho_{w_2}(x_0)}{\partial x^\beta}y'^\beta((D_1\psi)(y')-i(D_2\psi)(y'))$$
$$\cdot\lambda^{-(1-\delta)|\beta|+\beta_3(1-2\delta-\varepsilon)}$$

(6.13)
$$+\frac{1}{2}\sum_{\substack{0<|\beta|\le k\\ \beta_3\ge 1}}\frac{1}{\beta!}\frac{\partial^\beta\rho_{w_1}(x_0)}{\partial x^\beta}y'^\beta\psi(y')\lambda^{\delta-(1-\delta)|\beta|+\beta_3(1-2\delta-\varepsilon)}$$
$$+\frac{1}{2}\sum_{\substack{0<|\beta|\le k-1\\ \beta_3\ge 1}}\frac{1}{\beta!}\frac{\partial^\beta\rho_{w_1}(x_0)}{\partial x^\beta}y'^\beta(D_3\psi)(y')\lambda^{-(1-\delta)|\beta|+(\beta_3-1)(1-2\delta-\varepsilon)}$$
$$=-\tfrac{1}{4}(\partial\psi/\partial y_1'-i\partial\psi/\partial y_2')+O(\lambda^{-\varepsilon}),$$

where we have used (6.5).

3) Finally, the remainder yields

(6.14)
$$\sum_{|\alpha+\beta|\le k+1}y'^\beta(D^\alpha\psi)(y')\lambda^{\delta-(k+1)(1-\delta)-(\alpha_1+\alpha_2)(2\delta-1)+\beta_3(1-2\delta-\varepsilon)+\alpha_3\varepsilon}$$
$$=O(\lambda^{(k+1)\varepsilon+\delta-(k+1)(1-\delta)})=o(1),$$

where we have used (6.7). Thus (6.8), (6.9), (6.10), (6.12), (6.13) and (6.14) yield

(6.15)
$$\int_{R_3}|\psi(y)|^2dy\le\frac{1}{4}C\int_{R_3}\left|\frac{\partial\psi(y)}{\partial y_1}-i\frac{\partial\psi(y)}{\partial y_2}\right|^2dy,$$

where $\psi\in C_0^\infty(R_3)$. This is impossible. To see that set $\psi(y)=f(\varepsilon y)$, $f\in C_0^\infty(R_3)$, and let $\varepsilon\to 0$. Then the left hand side of (6.15) is $O(\varepsilon^{-3})$, while the right hand side is only $O(\varepsilon^{-2})$. Hence Theorem 3.7 is proved..

7. Remarks on the estimate (3.5)

In [8] Kohn proved that if $P\in\omega$ is of type m, and ω is pseudo-convex at P, then m must be odd. This result also follows by applying Propositions 2.4 of [9] to the symbol (5.7). Furthermore Kohn conjectured that under the hypothesis of Theorem 3.4 the estimate (3.5) holds with $s=1/(m+1)$.

7.1. Proposition. *Let $P\in\omega$ be a point of type m, and suppose that the estimate* (3.5) *holds with $s=1/(m+1)$. Then m is necessarily odd.*

Proof. It suffices to show that if the estimate (6.3) holds with $k = m, \delta = m/(m+1)$ and $(x_0, \xi^0) = (0,0,0,0,0,1)$, then m is odd. We shall follow the arguments of § 6 and indicate the necessary changes. Thus we substitute

$$\phi(y) = \psi(y_1, y_2, y_3\lambda^{2\delta-1+\varepsilon}) , \qquad \psi \in C_0^\infty(\boldsymbol{R}_3) \tag{7.2}$$

into (6.3), where

$$(m+1)\varepsilon + \delta < (m+1)(1-\delta) = 1 . \tag{7.3}$$

The left hand side of (6.3) again becomes (6.8). (6.9), (6.10) and (6.12) go through unchanged. (6.13) becomes

$$-\tfrac{1}{4}(\partial\psi/\partial y_1' - i\partial\psi/\partial y_2') + \tfrac{1}{2}\gamma_{w_1}(w_1, 0)\psi + O(\lambda^{-\varepsilon}) , \tag{7.4}$$

and there is no change in (6.14). Thus the hypothesis of Proposition 7.1 implies the following estimate

$$\int_{\boldsymbol{R}_3} |\psi(y)|^2 dy \leq C \int_{\boldsymbol{R}_3} \left| \frac{\partial\psi}{\partial y_1} - i\frac{\partial\psi}{\partial y_2} - 2\gamma_{w_1}(w_1, 0)\psi(y)\right|^2 dy , \tag{7.5}$$

where $\psi \in C_0^\infty(\boldsymbol{R}_3)$. Set

$$\psi(y_1, y_2, y_3) = \overline{f(y_1, y_2)} g(y_3) e^{2\gamma(w_1, 0)} .$$

Then (7.5) yields

$$\int_{\boldsymbol{R}_2} |f(y)|^2 e^{4\gamma(w_1,0)} dy \leq C \int_{\boldsymbol{R}_3} \left| \frac{\partial f}{\partial \bar{w}_1}\right|^2 e^{4\gamma(w_1,0)} dy , \tag{7.6}$$

for all $f \in C_0^\infty(\boldsymbol{R}_2)$. According to Theorem 2 of [2], (7.6) implies

$$\frac{\partial^2 \gamma(w_1, 0)}{\partial w_1 \partial \bar{w}_1} \geq 0 . \tag{7.7}$$

(Compare Kohn [8, formula (3.10)]. Egorov's proof of (7.7) is based on one of Hörmander's arguments in [4]; see [4, Lemma 1.2.4, especially (1.2.16)].) Now (7.7) clearly implies Proposition 7.1.

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